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GEOMETRY: EXAMPLES 2

1. Let Σ be the graph of a smooth function $F : \mathbb{R}^2 \to \mathbb{R}$. Compute its first fundamental form I, Gauss map n, second fundamental form II, and Gaussian curvature K.

2. Calculate K for the hyperboloid defined by $x^2 + y^2 - z^2 = c$, where $c \neq 0$. [*Hint: Parametrise it as a surface of revolution, after deleting the points* $(0, 0, \pm\sqrt{-c})$ *if* $c < 0$.] Describe the sign, behaviour near infinity, and c-dependence of K, and illustrate with diagrams for $c > 0$ and $c < 0$.

3. Suppose $\sigma(u, v)$ and $\tau(x, y)$ are overlapping parametrisations of an embedded surface Σ, with first fundamental form $E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 = E'\,\mathrm{d}x^2 + 2F'\,\mathrm{d}x\,\mathrm{d}y + G'\,\mathrm{d}y^2$.
   (a) Express the matrix $\left(\begin{smallmatrix} E' & F' \\ F' & G' \end{smallmatrix}\right)$ in terms of the matrix $\left(\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right)$, and show that you get the same answer if you set $\mathrm{d}u = u_x\,\mathrm{d}x + u_y\,\mathrm{d}y$ and $\mathrm{d}v = v_x\,\mathrm{d}x + v_y\,\mathrm{d}y$, and formally 'multiply out'.
   (b) Deduce that in polar coordinates the first fundamental form on $\mathbb{R}^2 \setminus \{0\}$ is $\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$.

4. (a) Show that inverse stereographic projection $\sigma^+ : \mathbb{R}^2 \to S^2$ is conformal, and deduce that the Earth (assumed to be a perfect sphere!) can be covered by maps which accurately represent angles.
   (b) Let C be the vertical cylinder defined by $x^2 + y^2 = 1$ and let $F : S^2 \setminus \{(0, 0, \pm 1)\} \to C$ be horizontal projection. Show that F is area-preserving and deduce that the Earth can be covered by maps which accurately represent areas up to a constant scale factor.
   *(c) Consider a parametrisation of $S^2 \setminus \{(0, 0, \pm 1)\}$ as a surface of revolution, i.e.
   $$\sigma(u, v) = \big(f(u)\cos v, f(u)\sin v, g(u)\big)$$
   with WLOG $g(0) = 0$ and $g'(0) > 0$. Show that there exists a unique choice of f and g such that σ is area-preserving. Show also that there exists a unique choice such that σ is conformal, but that there is no choice such that σ is a local isometry.

5. Show that any compact embedded surface Σ in $\mathbb{R}^3$ has an elliptic point, as follows. Let $p \in \Sigma$ be a point of maximal distance, R say, from the origin, and let $n(p)$ be the inward-pointing unit normal vector to Σ at p. Using the fact that Σ is contained in the sphere of radius R about the origin, and is tangent to it at p, show that for all $w \in T_p\Sigma$ we have
   $$\mathrm{II}_p(w, w) \geq \frac{1}{R}\mathrm{I}_p(w, w).$$
   [*Hint: points in* Σ *near* p *can be written as* $p + w + f(w)n(p)$, *for small* $w \in T_p\Sigma$, *and* II_p *is the Hessian of* f *at* 0.] By simultaneously diagonalising I_p and II_p, show that $K(p) \geq \frac{1}{R^2}$.

6. Fix a point p in an embedded surface Σ and recall from lectures that if $w \in T_p\Sigma$ is a unit vector then $\mathrm{II}_p(w, w)$ represents the curvature κ_w at p of the orthogonal slice through Σ in direction w.
   (a) Show that there exist real numbers k_1, k_2, such that κ_w is always between k_1 and k_2, and that κ_w attains these values in orthogonal directions w_1 and w_2. [*Hint:* $D_p n$ *is self-adjoint.*]
   (b) Explain why $k_1 k_2$ is a local isometry invariant, but show that k_1 and k_2 themselves need not be.
   (c) What implications does this have for how to pick up a slice of pizza?

7. Fix real numbers $a > b > c > 0$. For each of the following surfaces Σ compute the Gaussian curvature K and the integral $\int_\Sigma K\,\mathrm{d}A$.
   (a) The torus given by the image of $\sigma(u, v) = \big((a + b\cos u)\cos v, (a + b\cos u)\sin v, \sin u\big)$. (In lectures we have focused on the case $a = 2$, $b = 1$.)
   *(b) The ellipsoid defined by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$.

8. Let Σ be the cylinder defined by $x^2 + y^2 = 1$. Show that a geodesic on Σ through the point $(1, 0, 0)$ can be parametrized to be contained in a spiral of the form $\gamma(t) = (\cos\alpha t, \sin\alpha t, \beta t)$, where $\alpha^2 + \beta^2 = 1$.

9. Let $\Sigma \subset \mathbb{R}^3$ be a smooth embedded surface in $\mathbb{R}^3$.
   (a) Suppose that a straight line $\ell \subset \mathbb{R}^3$ lies entirely in Σ. Prove that ℓ is a geodesic on Σ.
   (b) Deduce that through every point p of the hyperboloid $x^2 + y^2 - z^2 = 1$ there are at least three complete geodesics.